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The Banach Contraction Principles Method for Approximate Solution of Sine-Gordon Equations

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ABSTRACT

This study systematically applies the Banach Contraction Principle Method (BCPM) to obtain highly accurate approximate solutions for the well-known nonlinear Sine-Gordon equation. The proposed analytical method is directly implemented by carefully reformulating the original boundary value problem into an equivalent integral form. Subsequently, we construct successive iterative approximations that mathematically converge to the exact analytical solution under suitable theoretical conditions. To thoroughly validate the accuracy and overall reliability of BCPM, comprehensive numerical examples are presented and examined in detail. The computational results demonstrate remarkably close agreement with exact solutions, particularly for relatively small-time intervals. Ultimately, the method demonstrates superior computational efficiency and numerical stability, even when operating with a significantly limited number of iterations. Consequently, this study establishes BCPM as a highly viable and practical tool for solving complex nonlinear partial differential equations.

1. INTRODUCTION

The Sine-Gordon equation is a partial differential equation that appears in various areas of physics, including field theory, soliton theory, and differential geometry. It is given by the equation:

$$u_{tt} - u_{xx} + \alpha \sin(u) = 0 \tag{1}$$

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with initial conditions

$$u(x,0) = g(x), \quad u_t(x,0) = h(x) \tag{2}$$

where, $u = u(x,t)$ is the dependent variable representing the field, t is time, x is the spatial variable of space and α is a constant parameter.

This equation intrinsically combines both wave-like and nonlinear behaviors, making it a subject of extensive study. The terms on the left side, specifically the second derivatives with respect to time and space, represent standard wave propagation and indicate how the field evolves dynamically. Conversely, the term on the right side is the nonlinear component, involving the sine function of the field variable u, which is responsible for complex physical phenomena such as the generation and interaction of solitary waves or solitons.

Due to the complex nonlinear nature of the Sine-Gordon equation, finding exact analytical solutions is often mathematically challenging, particularly under arbitrary initial or boundary conditions. Consequently, significant research has been directed toward developing efficient approximate and numerical techniques to solve such equations. Robust iterative methods that reformulate the differential problem into an integral form are highly desirable, as they offer stable and reliable ways to capture the intricate dynamics of the system.

While various numerical and analytical methods such as the Adomian Decomposition Method (ADM) and Variational Iteration Method (VIM) have been employed to solve the Sine-Gordon equation, they often involve complex symbolic computations or the determination of Lagrange multipliers. The novelty of this study lies in the strategic implementation of the Banach Contraction Principle Method (BCPM) as a direct iterative scheme. Unlike existing methods, BCPM bypasses the need for calculating Adomian polynomials or restrictive transformation steps, offering a more straightforward and computationally efficient framework. This study provides a new perspective on how fixed-point theorems can be utilized not just for existence proofs, but as a robust practical tool for generating high-precision approximate solutions for nonlinear wave equations.

2. LITERATURE REVIEW

The Sine-Gordon equation, a fundamental nonlinear partial differential equation, has been the subject of extensive research due to its applications in soliton theory, field theory, and differential geometry. Over the past decade, various analytical and numerical methods have been developed for its solution.

Studies focusing on exact and analytical solutions include the work of Johnson et al. (2012), who derived new exact solutions for the $2+1$-dimensional Sine-Gordon equation, highlighting its relevance to two-dimensional soliton dynamics. Wazwaz (2014) employed Hirota's method to construct n-soliton solutions for multi-dimensional forms of the equation, identifying specific conditions for soliton existence. More recently, Sadiya et al. (2022) applied an extended tanh-function approach to obtain traveling wave solutions for fractional versions of the Sine-Gordon equation.

In the realm of numerical and semi-analytical approaches, Pekmen and Tezer-Sezgin (2012) utilized the Differential Quadrature method to solve the 2D Sine-Gordon equation, demonstrating improved accuracy with increased spatial discretization. Ali et al. (2022) applied the Laplace Adomian Decomposition Method to the fractal-fractional Sine-Gordon equation, achieving series solutions that converge effectively. Batiha (2022) introduced the Daftardar-Gejji and Jafari method as a new solution approach, comparing its performance favorably with the Variational Iteration Method.

Despite these advances, the BCPM has not been widely applied to the Sine-Gordon equation. BCPM is a robust iterative technique grounded in fixed-point theory, proven effective for a wide range of nonlinear

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problems. Its foundational applications include solving multivalued monotone variational inequalities (Anh et al., 2005) and nonlinear functional equations (Daftardar-Gejji & Bhalekar, 2009). Subsequent developments have broadened its scope. Proinov (2007) generalized the method with high-order convergence, while Al-Jawary et al. (2018) successfully adapted it to model non-Newtonian fluid flows, achieving accurate results with fewer iterations than conventional methods.

The versatility of BCPM is further evidenced by its successful application to fractional integro-differential equations (Işık et al., 2019), dimensional analysis in wave equations (Adwan et al., 2020), the Falkner-Skan equation (Al-Jawary & Adwan, 2020), the SIR epidemic model (Abed & Al-Jawary, 2021), and the Drinfeld–Sokolov–Wilson system (Raslan & Entesar, 2022). These studies collectively affirm BCPM as a reliable, efficient, and precise iterative method for nonlinear problems. This establishes a strong rationale for its novel application to the Sine-Gordon equation, which is the central aim of the present study.

The versatility of iterative methods, including BCPM and its related approaches, is further evidenced by their successful application across diverse mathematical and epidemiological models. For instance, recent studies have demonstrated the robustness of the integral iterative method in solving the Newell-Whitehead-Segel equation (Selamat et al., 2022) and the Picard iterative method in modeling Monkeypox transmission dynamics (Khairuddin et al., 2025). Similarly, BCPM has been successfully applied to fractional integro-differential equations (Işık et al., 2019), dimensional analysis in wave equations (Adwan et al., 2020), the Falkner-Skan equation (Al-Jawary & Adwan, 2020), the SIR epidemic model (Abed & Al-Jawary, 2021), and the Drinfeld–Sokolov–Wilson system (Raslan & Entesar, 2022). These studies collectively affirm these mathematical approaches as reliable, efficient, and precise iterative methods for nonlinear problems. This establishes a strong rationale for its novel application to the Sine-Gordon equation, which is the central aim of the present study.

3. METHODOLOGY

This section presents the methodology for solving the Sine-Gordon equation. It covers the essential mathematical preliminaries, introduces the theoretical framework of the Banach Contraction Principle Method (BCPM), and details its direct implementation to derive approximate solutions.

3.1 Preliminaries

This section begins with some basic concepts:

Definition: A mapping F from metric space X_1 into metric space X_2 is said to be a Lipschitz mapping if there exists a real number $r \geq 0$ such that for all $x_1, x_2 \in X_1$, Then obtain $d(Fx_{1,}Fx_2) \leq d(x_1, x_2)$, where $d(x, y)$ represents the distance between points $x$and y in the metric space (Joshi & Bose, 1985).

Theorem 1 (Banach Fixed-Point Theorem): F is a contraction mapping with Lipschitz constant r on a complete metric space X into itself, then F has a unique fixed point V in the space X. In addition, if x_0 is an arbitrary point in X, and the sequence, $\{x_n\}$ is defined by $x_{n+1} = F(x_n)$, $n = 0, 1, 2, \dots$ then, $\lim_{n \to \infty} x_n = V$. Additionally, the rate of convergence is exponential, and $d(x_n, V) \leq \frac{r^n}{1-r} d(x_1, x_0)$ (Joshi & Bose, 1985).

Theorem 2: Let F be a mapping of some complete metric space X into itself, such that F^k is a contraction mapping of X for a positive integer k, then F has a unique fixed point in the space X (Joshi & Bose, 1985).

Theorem 2 is called the Banach Fixed Point Theorem for iterated contraction.

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3.2 Banach Contraction Principles Method

Consider the general nonlinear functional equation (Daftardar-Geiji & Bhalekar, 2009),

$$u(x) = f(x) + N[u(x)], \tag{3}$$

where $u(x)$ is an unknown function, $f(x)$ is a given function and N is a nonlinear operator of the functional equation (3). Next, the successive approximation defined as

$$\begin{aligned} u_0 &= f, \\ u_1(x) &= u_0(x) + N[u_0(x)] \\ u_2(x) &= u_0(x) + N[u_1(x)], \\ &\vdots \\ u_{m+1}(x) &= u_0(x) + N[u_m(x)], \qquad m = 0,1,2,3,\ldots \end{aligned} \tag{4}$$

If N^k is the contraction for some positive integer, k then $N(u)$ has a unique fixed point and hence the sequence defined by (4) is convergent in view of Theorem 2 and the solution of (3) is given by:

$$u = \lim_{n\to\infty} u_n \tag{5}$$

3.3 Implementation of BCPM on the Sine-Gordon Equation

The implementation of the BCPM for the Sine-Gordon equation requires a systematic transformation from a differential form into an integral operator. Consider the Sine-Gordon equation (1) with initial conditions (2). Equation (1) can be expressed as

$$u_{tt} = u_{xx} - \alpha \sin(u) \tag{6}$$

To transform this differential form into an integral equation, a double integration operator with respect to t is applied to both sides over the interval $[0, t]$. This process incorporates the initial conditions $u(x, 0)$ and $u_t(x, 0)$ to account for the system's initial state. By isolating the displacement variable u, the following integral form is obtained: In view of the BCM, integrating both sides of (6) from 0 to t and yields:

$$u = \iint_0^t (u_{xx} - \alpha \sin(u))dtdt \tag{7}$$

The initial approximation u_0 is defined by the given initial functions. Following the successive approximation property of the Banach Fixed-Point Theorem, each subsequent iteration u_{n+1} is generated by substituting the previous term u_n into the nonlinear integral operator. The initial approximation u_0 can be written as:

$$u_0 = \iint_0^t 0dt = f(x). \tag{8}$$

The first iteration u_1 can be written as:

$$u_1 = u_0 + \iint_0^t (u_{0xx} + \alpha \sin(u_0))dtdt \tag{9}$$

Similarly, the second approximation can be read as:

$$u_2 = u_0 + \iint_0^t (u_{1xx} + \alpha \sin(u_1))dtdt \tag{10}$$

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Therefore, in general, approximation can be expressed as:

$$u_{n+1} = u_0 + \iint_0^t (u_{nxx} + \alpha \sin(u_n))dtdt \tag{11}$$

4. NUMERICAL APPLICATIONS

In this section, two numerical examples are solved by BCPM to reveal the reliability and accuracy of the method.

4.1 Example 1

Consider the pendulum-like equation (Batiha, 2022):

$$u_{tt} = \sin t \tag{12}$$

with initial condition

$$u(x,0) = \pi, \quad u_t(\mathrm{x},0) = -2 \tag{13}$$

The derivation of the initial approximation u_0 is performed by integrating the initial velocity $u_t(x,0) = -2$ and adding the initial displacement $u(x,0) = \pi$. This results in the linear expression of $u_0 = \pi - 2t$. Where the exact solution for (12) is

$$u(x,t) = 2\sin^{-1}(\operatorname{sech}^{-1} t) \tag{14}$$

By using BCPM, the equivalent integral equation of (12) is

$$u = u_0 + \iint_0^t \sin t \, dtdt \tag{15}$$

Thus, the first few iterative solutions are,

$$u_0 = \pi - 2t \tag{16}$$

$$u_1 = \frac{3t}{2} + \pi - \frac{1}{4}\sin 2t \tag{17}$$

The calculation stopped at u_1 as BCPM precision agrees with the exact solution to at least 7 decimal places. Table 1 presents a comparison between the obtained approximation values using the BCPM and the corresponding exact values for the Sine-Gordon equation at different time points within the interval. The calculated results were obtained through the utilization of Maple software, enabling $t = [0,1]$ the determination of approximate values, exact values, and the absolute errors associated with each approximation.

Table 1. Error between the present solution with the exact solution for $t = [0,1]$.

t	BCPM, u_1	Exact value	Absolute error
0.0	3.141592654	3.141592654	0
0.1	2.941925321	2.941925156	1.6530×10^{-7}
0.2	2.744238068	2.744232960	5.10842×10^{-6}
0.3	2.550432036	2.550395302	0.0000367336
0.4	2.362253631	2.362110412	0.0001432193
0.5	2.181224908	2.180830496	0.0003944118
0.6	2.008582882	2.007721526	0.0008613565
0.7	1.845230222	1.843648490	0.0015817315
0.8	1.691699253	1.689183008	0.0025162452
0.9	1.548130746	1.544628063	0.0035026833
1.0	1.414268297	1.410053687	0.0042146103

Table 1 shows that the BCPM approximation values align with the exact values across the range of $0.0 \leq t \leq 1.0$. It is evident that as t progressed from 0.0 to 1.0, both the approximation values and the exact values changed accordingly. The absolute error column indicates the discrepancy between the approximation and the exact value at each time step.

Upon analyzing the results, it's noticeable that the approximation values obtained using BCPM converged towards the exact values for the given initial conditions. However, as t increased, the absolute error also increased. This phenomenon is attributed to the sensitivity of the power series and the nature of the solution to the parameters inherent in the series. In simpler terms, as the solution point moves further away from the initial starting point, the accuracy of the approximation becomes less precise.

4.2 Example 2

Now examined the Sine-Gordon equation (1) with initial conditions (Batiha, 2022):

$$u(x,0) = 0, \quad u_t(\mathrm{x},0) = 4\ \operatorname{sech} x \tag{18}$$

where the exact solution is

$$u(x,t) = 4\tan^{-1}(t \operatorname{sech} x) \tag{19}$$

By using BCPM, yields:

$$u_0 = 4t \operatorname{sech} x \tag{20}$$

$$u_1 = 4ts - \frac{[-64t^3s^3\,(\tanh x)^2 + 32t^3s^3 + 12\cosh(x)\,s^2t - 3\sin\,(4ts)}{48s^3} \tag{21}$$

where $s = \operatorname{sech} x$.

Table 2. Absolute error between exact solution with BCPM results for $t = 1, 0.1, 0.01$.

x	$t = 1$	$t = 0.1$	$t = 0.01$
0	0.1055594766	2.630271×10^{-6}	2.9166×10^{-11}
0.1	0.0998890218	2.513742×10^{-6}	2.4238×10^{-11}
0.2	0.0839187011	2.187760×10^{-6}	2.3116×10^{-11}
0.3	0.0604765632	1.714742×10^{-6}	1.3719×10^{-11}
0.4	0.0334099755	1.177308×10^{-6}	4.485×10^{-12}
0.5	0.0066362251	6.549530×10^{-7}	1.00828×10^{-11}
0.6	0.0166687163	2.070947×10^{-7}	4.8953×10^{-12}
0.7	0.0345318512	1.342186×10^{-7}	4.130×10^{-13}
0.8	0.0462513575	3.625622×10^{-7}	2.4613×10^{-12}
0.9	0.0521654534	4.898078×10^{-7}	4.9128×10^{-12}
1.0	0.0532765167	5.375046×10^{-7}	5.1530×10^{-12}

Table 2 provides an overview of the absolute error between the exact solution and the results obtained using the BCPM for varying values of t and x. The table presents the absolute error values at specific time points: $t = 1,\ t = 0.1$, and $t = 0.01$. Each row in the table corresponds to a different value of x, and each column represents a different time point.

The x column signifies the spatial coordinate, indicating the position along the x-axis. The rows beneath each time point provide the absolute error values associated with that specific x value and time point. The absolute error quantifies the discrepancy between the exact solution and the BCPM-generated results at the given time and position.

When analyzing the table, a few observations had been made. Firstly, as the time interval becomes smaller (from $t = 1$ to $t = 0.01$), the absolute error generally decreases. This behavior is expected, as a smaller time interval allows for more accurate approximation and results in less deviation from the exact solution.

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Secondly, for each time point, as x increases, the absolute error tends to increase as well. This trend indicates that the accuracy of the BCPM results diminishes as the position x moves farther from the initial reference point.

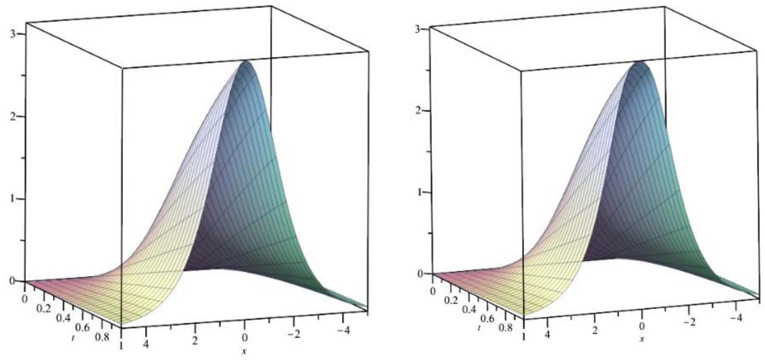

Fig. 1. (left) one iteration of BCM; (right) the exact solution.

Fig. 1 (left) shows the approximate solution of the sine-Gordon equation using the Banach Contraction Method (BCM), while Fig. 1 (right) represents the exact solution. Visually, both solutions exhibit a nearly identical surface structure: a smooth, localized wave that grows over time and decays symmetrically in the spatial variable x. The peak values and overall shape match closely, indicating that the BCM approximation accurately captures the amplitude and dynamics of the exact solution. There are no visible discrepancies in curvature, smoothness, or rate of change, suggesting high accuracy of the numerical method within the plotted domain. Hence, the BCM provides an excellent approximation to the exact solution, demonstrating both accuracy and reliability.

4.3 Example 3

This example reconsidered the Sine-Gordon equation (1) with initial conditions (Batiha, 2022):

$$u(x,0) = \pi + \gamma\ \cos\beta x, \quad u_t(\mathrm{x},0) = 0 \tag{22}$$

where γ is any constant and $\beta = \sqrt{2}/2$. By employing BCPM, obtained

$$u_0 = \pi + \gamma\cos\beta x \tag{23}$$

$$u_1 = \pi + \gamma\cos\beta x - \frac{1}{2}(\gamma\cos(\beta x)\beta^2 - \sin(\gamma\cos(\beta x)))^2 \tag{24}$$

which are the same results as Daftardar-Gejji and Jafari method reported by Batiha (2022).

To show the stability of BCPM results, three γ values ($\gamma = 0.001, 0.05$ and 0.1) were chosen and plot the graph in Fig. 2. The solution to the Sine-Gordon equation shown in Figure 2 is considered stable because it remains smooth, bounded, and oscillates regularly around $x = \pi$ without any sign of growth, decay or chaotic behavior over time. This indicates that small perturbations do not amplify, and the solution retains its structure across both space and time. Such behavior is characteristic of stable solutions, particularly solitons or breathers, which are well-known in sine-Gordon theory to exhibit robustness under evolution. The confined and consistent nature of the surface plot supports both mathematical and numerical stability.

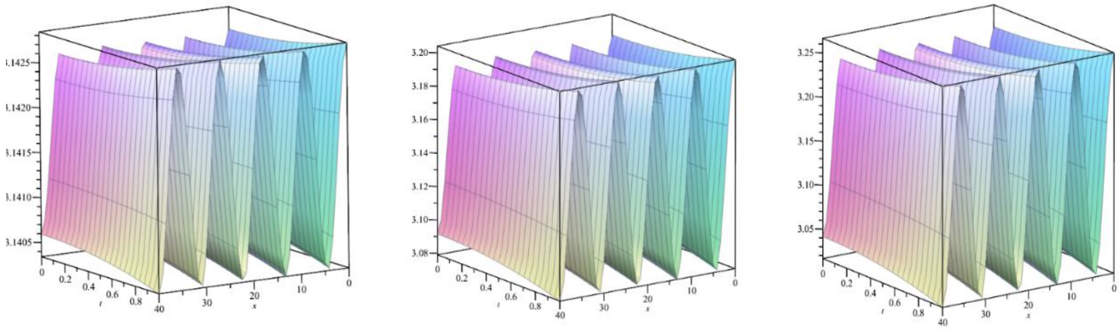

Fig. 2. The numerical results for the Sine-Gordon equation using/employing one iteration of the BCM for (left) $\gamma = 0.001$; (middle) $\gamma = 0.05$; (right) $\gamma = 0.1$.

The performance of the BCPM is significantly highlighted by its rapid convergence and high precision. In the examples 1, the absolute error obtained is as low as 10^{-7} after only a single iteration, which demonstrates a superior result compared to several existing numerical schemes. While traditional methods such as the ADM or VIM often involve the calculation of complex Adomian polynomials or the determination of Lagrange multipliers to achieve similar accuracy, the BCPM achieves these results through a direct and more straightforward iterative process. This comparison confirms that the BCPM is a highly efficient approach for the Sine-Gordon equation, offering a notable reduction in computational complexity while maintaining exceptional numerical stability.

5. CONCLUSION

This study focused on solving the nonlinear Sine-Gordon equation using the Banach Contraction Principle Method (BCPM). The primary aim was to apply the BCPM to solve homogeneous examples of the Sine-Gordon equation with specific initial conditions, and this objective was successfully accomplished. By implementing the BCPM, the study achieved its goal of generating accurate approximate solutions. The results obtained through this method were compared with exact solutions, revealing that the BCPM yields results with remarkable precision that closely align with exact values. Specifically, the accuracy is enhanced when the power series utility range converges, resulting in minimal error values that approach zero.

Based on these findings, it is evident that the BCPM exhibits significant effectiveness and precision when applied to the Sine-Gordon equations in the selected examples. This work contributes a robust and straightforward numerical framework for nonlinear PDEs, demonstrating that BCPM is not only simpler to implement eliminating the need for complex Adomian polynomials or Lagrange multipliers but also highly efficient in maintaining superior numerical stability with a limited number of iterations.

Despite its effectiveness, this study is subject to certain limitations. The current implementation focuses primarily on one-dimensional Sine-Gordon equations within relatively small-time intervals. As the time interval t increases, the convergence rate may require a higher number of iterations to maintain the desired precision. Additionally, the study does not account for external damping factors or stochastic variations that might be present in real-world physical systems.

For future research, it is suggested that the BCPM be extended to solve two-dimensional or three-dimensional Sine-Gordon models. Further investigations could also explore the application of this method to fractional-order Sine-Gordon equations or its integration with other numerical techniques to enhance performance over larger time domains.

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7. CONFLICT OF INTEREST STATEMENT

The authors declare there is no conflict of interest in the subject matter or materials discussed in this manuscript.

8. AUTHORS' CONTRIBUTIONS

Mat Salim Selamat carried out the research, wrote and revised the article. Ahmad Bazli Khairuddin, Nurain Nabila Hamzah, and Siti Hidayah Muhad Saleh performed the numerical computations and validated the analytical results. Rosha Mohamed and Rahmah Shahril provided the theoretical framework and assisted in the data analysis. Busyra Latif conceptualised the central research idea, supervised research progress, anchored the review, revisions and approved the article submission.

REFERENCES

Abed, S. M., & Al-Jawary, M. A. (2021). Efficient iterative methods for solving the SIR epidemic model. *Iraqi Journal of Science*, *62*(2), 613–622. https://doi.org/10.24996/ijs.2021.62.2.27

Adwan, M. I., Al-Jawary, M. A., Tibaut, J., & Ravnik, J. (2020). Analytic and numerical solutions for linear and nonlinear multidimensional wave equations. *Arab Journal of Basic and Applied Sciences*, *27*(1), 166–182. https://doi.org/10.1080/25765299.2020.1751439

Ali, A., Khan, A. U., Algahtani, O., & Saifullah, S. (2022). Semi-analytical and numerical computation of fractal-fractional Sine-Gordon equation with non-singular kernels. *AIMS Mathematics*, *7*(8), 14975–14990. https://doi.org/10.3934/math.2022820

Al-Jawary, M. A., & Adwan, M. I. (2020). Reliable iterative methods for solving the Falkner–Skan equation. *Gazi University Journal of Science*, *33*(1), 168–186. https://doi.org/10.35378/gujs.457840

Al-Jawary, M. A., Radhi, G. H., & Ravnik, J. (2018). Development of the Banach contraction method for the solution of nonlinear thin film flows of non-Newtonian fluids. *Arab Journal of Basic and Applied Sciences*, *25*(3), 122–131. https://doi.org/10.1080/25765299.2018.1511079

Anh, P. N., Muu, L. D., Nguyen, V. H., & Strodiot, J. J. (2005). Using the Banach contraction principle to implement the proximal point method for multivalued monotone variational inequalities. *Journal of Optimization Theory and Applications*, *124*(2), 285–306. https://doi.org/10.1007/s10957-004-0926-0

Batiha, B. (2022). New solution of the Sine-Gordon equation by the Daftardar-Gejji and Jafari method. *Symmetry*, *14(1)*, Article 57. https://doi.org/10.3390/sym14010057

Daftardar-Gejji, V., & Bhalekar, S. (2009). Solving nonlinear functional equation using Banach contraction principle. *Far East Journal of Applied Mathematics*, *34*(3), 303–314.

Işık, H., Mohammadi, B., Haddadi, M. R., & Parvaneh, V. (2019). On a new generalization of Banach contraction principle with application. *Mathematics,* *7*(9), 862. https://doi.org/10.3390/math7090862

Johnson, S., Suarez, P., & Biswas, A. (2012). New exact solutions for the sine-Gordon equation in 2+1 dimensions. *Computational Mathematics and Mathematical Physics*, *52*, 98–104. https://doi.org/10.1134/S0965542512010058

Joshi, M. C., & Bose, R. K. (1985). *Some topics in nonlinear functional analysis*. Wiley.

Khairuddin, A. B., Selamat, M. S., & Yacob, N. A. (2025). Solution of Monkeypox transmission model using Picard Iterative Method. *Mathematical Sciences and Informatics Journal*, *6*(2), 43–59. https://doi.org/10.24191/mij.v6i2.4652

Pekmen, B., & Tezer-Sezgin, M. (2012). Differential quadrature solution of nonlinear Klein-Gordon and Sine-Gordon equations. *Computer Physics Communications*, *183*(8), 1702–1713. https://doi.org/10.1016/j.cpc.2012.03.010

Proinov, P. D. (2007). A generalization of the Banach contraction principle with high order of convergence of successive approximations. *Nonlinear Analysis: Theory, Methods & Applications*, *67*(8), 2361–2369. https://doi.org/10.1016/j.na.2006.09.008

Raslan, A., & Entesar, A. (2022). Enhancing Banach's contraction method using the particle swarm optimization to solve the system Drinfeld–Sokolov–Wilson. *Journal of Physics: Conference Series*, *2322*, 012031. https://doi.org/10.1088/1742-6596/2322/1/012031

Sadiya, U., Inc, M., Arefin, M. A., & Uddin, M. H. (2022). Consistent travelling waves solutions to the non-linear time fractional Klein–Gordon and Sine-Gordon equations through extended tanh-function approach. *Journal of Taibah University for Science*, *16*(1), 594-607. https://doi.org/10.1080/16583655.2022.2089396

Selamat, M. S., Khusairi, N. A., Abdul Gaffar, N. I., & Syed Huzaini, S. N. S. (2022). The integral iterative method for approximate solution of Newell-Whitehead-Segel equation. *Mathematical Sciences and Informatics Journal*, *3*(1), 1–10. https://doi.org/10.24191/mij.v2i2.13009

Wazwaz, A. M. (2014). N-soliton solutions for the integrable modified KdV-sine-Gordon equation. *Physica Scripta*, *89*(6), 065805. https://doi.org/10.1088/0031-8949/89/6/065805

https://doi.org/10.24191/mij.v7i1.9787